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Distribution of distance in the spheroid

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Abstract
The distribution of distance in the sphere is reviewed. The distribution of distance in the ellipsoid is given as an integral which can be done in terms of elementary functions for the spheroid. As an application, Maclaurin's ratio of the polar to equatorial radius of the Earth due to its rotation is rederived using the distribution found here.

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1. Introduction

In physics, the potential energy of a body is, very often, the sum of the potential energies of its pairs of parts or particles. From a computational point of view, this means that we have to do a six-dimensional integral. But if the distribution of distance of the body is known then only a one-dimensional integral is necessary. This increases the numerical precision by many orders of magnitude and allows the computation of potential energies in instances where a straightforward approach would be too costly. Applications to nuclear physics and electrostatics based on this idea have been discussed recently [1–4]. Some of these authors have also applied the distribution of distance to the testing of random number generators [5, 4].

Computation of potential energy is the main application of the distribution of distance to physics, but not the only one, the analysis of mobile radio systems [6] being another.

The topic has, of course, intrinsic mathematical interest and has been attracting mathematicians for some time. There is a branch of mathematics which is, loosely speaking, an outgrowth of Buffon's needle problem (1777) and Bertrand's paradox (1907) and which is called integral geometry or geometric probability [7–9]. This branch, which attracted the attention of Poincaré and Mark Kac (see the foreword to the book by Santaló [8]), has dealt with the problem of distribution of distance.

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A number of authors have found the distribution of distance of some geometrical figures starting, to the best of our knowledge, in 1919 [10], when the distribution of distance for the sphere was found.

Other references [11–21], as well as a historical overview of the problem of distribution of distance (with especial emphasis on its applications to geography) are given in the doctoral dissertation of de Smith [22]. These references, as can be read in their titles, cover the rectangle, the n-dimensional sphere, concentric shells of different densities [1, 5] and the triangle [15]. Moments of distance distributions are found in [5] as well as in an interesting web page on the subject [23].

In this paper, we rederive, for the sake of self-containedness, the distribution of distance in the sphere (section 2). New results are: (1) an integral expression for the distribution of distance of the ellipsoid (section 3); (2) an expression in terms of elementary functions for an ellipsoid two of whose axes have equal length (also called 'spheroid') (section 4); (3) as an application, a result on the shape of the Earth, obtained by Maclaurin in 1732, is derived here in a new way (section 5).

It should be stressed that the distribution of distances in a geometrical body is directly relevant to a physical body only under the assumption that the latter has homogeneous density. As explained in the last paragraph of section 3, the results in this paper are also pertinent to some inhomogeneous distributions of physical density.

2. Spheres

2.1. Discs

The probability density $\rho(\ell)$ for the distance between two randomly chosen points inside a domain D to be ℓ is

$$\rho(\ell) = \frac{\int_D \mathrm{d}\vec{y} \int_D \mathrm{d}\vec{x}\, \delta(|\vec{x} - \vec{y}| - \ell)}{\int_D \mathrm{d}\vec{y} \int_D \mathrm{d}\vec{x}}. \tag{1}$$

In two dimensions, the numerator can also be written as follows. Consider a circumference of radius ℓ about $\vec{x}$, where $\vec{x}$ lies in D. Let us denote by $O(\vec{x}, \ell)$ the length of its overlap with D. Then the numerator can be written as

$$\int_D \mathrm{d}\vec{x}\, O(\vec{x}, \ell). \tag{2}$$

We have obtained the above expression by a two-dimensional argument, but it holds for any dimension n, provided that the word 'circumference' is substituted by '$(n-1)$-dimensional spherical shell' and the word 'length' by '$n-1$ area'.

For a disc of radius 1, $\int_D \mathrm{d}\vec{x}\, O(\vec{x}, \ell) = 2\pi \int_0^1 \mathrm{d}x\, x\, O(x, \ell)$, where $x \equiv |\vec{x}|$. It is clear that, for $x + \ell < 1$, $O(x, \ell) = 2\pi\ell$ (the smaller circle of figure 1). When $\ell < 1$, there are two other cases (we take ℓ to be fixed and let x vary). In the first (figure 2(a)), it is the largest piece of the circumference of radius ℓ that is $O(x, \ell)$; in the second (figure 2(b)) it is the smallest piece. In either case the length of the circumference of radius ℓ about $\vec{x}$ which is inside the disc is $2\left(\pi - \arccos \frac{1-x^2-\ell^2}{2x\ell}\right)$, provided that we choose the arccos function whose range is $[0, \pi]$. Taking into account that the denominator in equation (1) is simply the square of the area, π^2, we can now compute ρ for the disc:

$$\rho(\ell) = \frac{2\pi}{\pi^2} \int_0^1 \mathrm{d}x\, x\, O(x, \ell) = 4\ell \int_0^{1-\ell} \mathrm{d}x\, x + \frac{4\ell}{\pi} \int_{1-\ell}^1 \mathrm{d}x\, x \left(\pi - \arccos \frac{1 - x^2 - \ell^2}{2x\ell}\right). \tag{3}$$

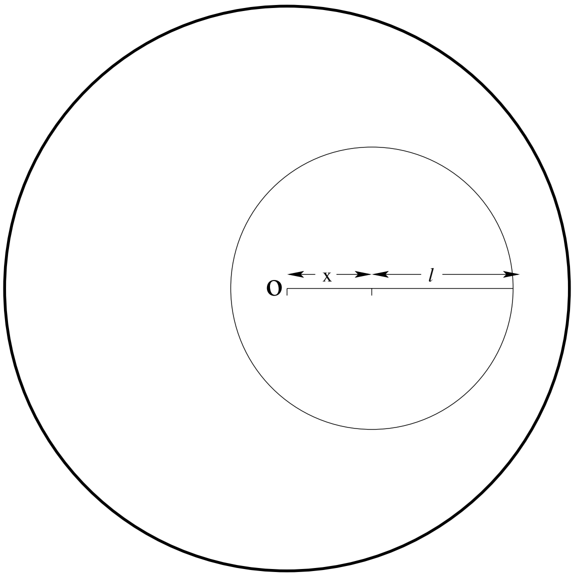

Figure 1. The case $x + \ell < 1$.

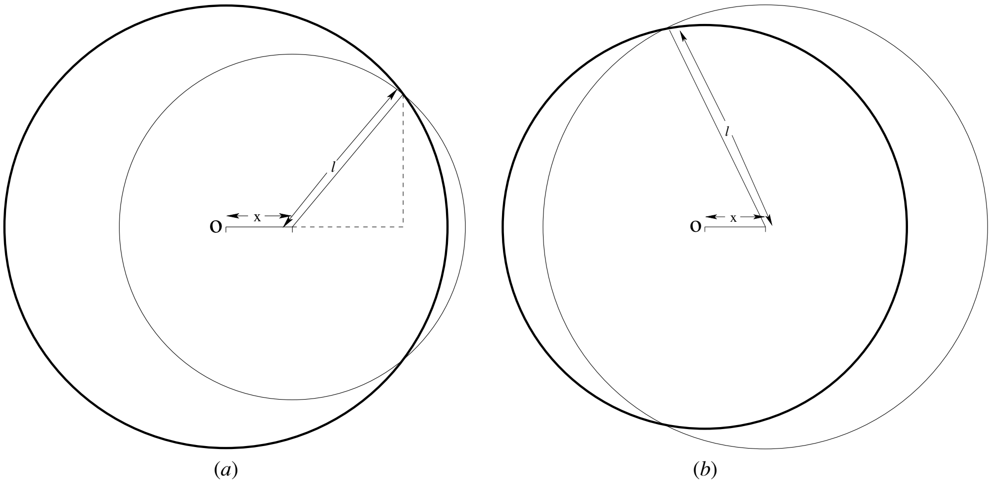

Figure 2. (*a*) The longest piece of the circumference of radius ℓ is the one that is $O(x, \ell)$. (*b*) The shortest piece of the circumference of radius ℓ is the one that is $O(x, \ell)$.

The first integral is straightforward and the second can be done by parts:

$$\rho(\ell) = \frac{4\ell}{\pi} \arccos\frac{\ell}{2} - \frac{2\ell^2}{\pi}\sqrt{1 - \frac{\ell^2}{4}}. \quad (4)$$

The result for a disc of radius r follows from dimensional analysis:

$$\rho(\ell) = \frac{4\ell}{\pi r^2} \arccos\frac{\ell}{2r} - \frac{2\ell^2}{\pi r^4}\sqrt{r^2 - \frac{\ell^2}{4}}. \quad (5)$$

In figure 3, we have plotted the probability density ρ. It is a bell-shaped curve whose mode is twice the root of $\arccos x = 3x\sqrt{1 - x^2}$ ($\approx 0.836\,222$) and mean $\frac{128}{45\pi} \approx 0.9054$.

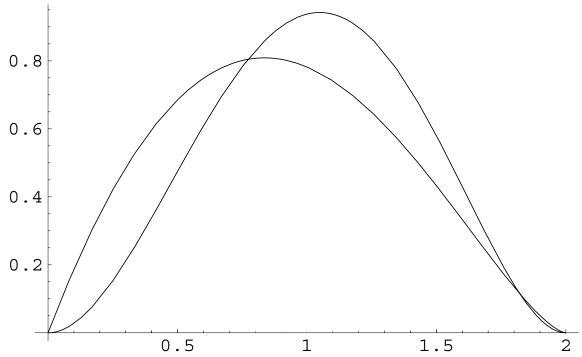

Figure 3. Probability densities for the disc (the fatter one) and the sphere (the taller one). The horizontal axis goes from 0 to 2, the range of distances in the unit disc or sphere.

2.2. *Spheres*

For the sphere we have to imagine that figures 1, 2(*a*) and (*b*) revolve around the horizontal diameter. Now we want to find not the length of the arc of radius ℓ, but the surface of the corresponding spherical cap. The area of a spherical outside of the sphere can be readily found to be $2\pi(1 - \cos\arccos\frac{1-x^2-\ell^2}{2xd}) = 2\pi(\frac{(x+\ell)^2-1}{2x\ell})$. We proceed as in the disc:

$$\begin{aligned}\rho(\ell) &= \frac{4\pi}{(4\pi/3)^2}\int_0^1 \mathrm{d}x\, x^2 O(x,\ell) \\ &= \frac{4\pi}{(4\pi/3)^2}\int_0^{1-\ell} \mathrm{d}x\, x^2 4\pi\ell^2 + \frac{4\pi}{(4\pi/3)^2}\int_{1-\ell}^1 \mathrm{d}x\, x^2\left(4\pi - 2\pi\left(\frac{(x+\ell)^2-1}{2x\ell}\right)\ell^2\right) \\ &= \frac{3}{16}(\ell-2)^2\ell^2(\ell+4). \end{aligned} \tag{6}$$

Early references on this result are [10–12, 14]. This probability density is plotted in figure 3. Its mode, $\frac{\sqrt{105}}{5} - 1 = 1.049\ldots$ is now greater than its mean, $\frac{36}{35} = 1.028\,571\,428\,5714\ldots$.

3. The ellipsoid: a geometrical derivation

An ellipse is nothing but a circle which has been stretched by a factor (e.g., λ) along some direction (e.g., the x direction). Consider any given distance ℓ. In the circle, ℓ appears as the distance between, say, the points (x, y) and (x', y'), which is $\sqrt{(x-x')^2 + (y-y')^2}$. Upon stretching this distance becomes $\sqrt{\lambda^2(x-x')^2 + (y-y')^2}$. Due to the symmetry of the circle, the distance ℓ will thus transform into the distance $\ell\sqrt{(\sin\alpha)^2 + \lambda^2(\cos\alpha)^2}$ with probability density $2/\pi$. If we define $f_\lambda(\alpha) = \sqrt{(\sin\alpha)^2 + \lambda^2(\cos\alpha)^2}$, then the probability density that ℓ is the distance between two randomly chosen points of an ellipse of semiaxes 1 and λ is

$$\rho_\lambda(\ell) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \mathrm{d}\alpha \int_0^2 \mathrm{d}x\, \rho(x)\delta(x f_\lambda(\alpha) - \ell) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \mathrm{d}\alpha \frac{1}{f_\lambda(\alpha)}\rho\left(\frac{\ell}{f_\lambda(\alpha)}\right), \tag{7}$$

where ρ is the distances' probability density of the unit circle found earlier (equation (5)).

One can repeat the above argument in a higher dimension and obtain the distances' probability density for an ellipsoid of semiaxes 1, λ and μ. Due to the relative simplicity and importance in gravitation and nuclear physics of this formula, we write it in detail:

$$\rho_{\lambda,\mu}(\ell) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta \sin\theta \int_0^{\frac{\pi}{2}} d\varphi \frac{1}{\sqrt{(\lambda^2\cos^2\varphi + \mu^2\sin^2\varphi)\sin^2\theta + \cos^2\theta}} \times \rho\left(\frac{\ell}{\sqrt{(\lambda^2\cos^2\varphi + \mu^2\sin^2\varphi)\sin^2\theta + \cos^2\theta}}\right), \tag{8}$$

where

$$\rho(x) = \frac{3}{16}(x-2)^2x^2(x+4), \tag{6}$$

for x in [0, 2], and 0 for x outside it (do not forget this when substituting in the expression for $\rho_{\lambda,\mu}(\ell)$!). Formula (8), though not exactly simple, should be compared with its numerical alternative, which involves sampling over a six-dimensional space (the Cartesian product of two ellipsoids). The integrand of formula (8) is more complicated than the distance, which would be the 'integrand' in the straightforward numerical method. However, for any meaningful precision, the fact that the above expression involves a two-dimensional integral, as opposed to a six-dimensional one, makes it numerically much more efficient.

Actually, formula (8) can be integrated over θ to yield a one-dimensional integral of elementary functions, but we think that it would be lengthy, but not illuminating, to write it.

It is important to realize that formulae (7) and (8) give the probability density of distance, $\rho_{\lambda,\mu}$, of any distribution of physical density with ellipsoidal symmetry provided that its corresponding spherical probability density of distance, ρ, is known. We have focused here in the case in which ρ is the probability density of distance for the homogeneous sphere, but the formulae can be applied to other spherically symmetric ρs which have been studied: the case of an isotropic Gaussian [1, 5] and the case of a sphere made of spherical shells of different densities [1, 5].

4. Distribution of distance in the spheroid

For the case $\mu = \lambda$ (called the 'spheroid') an expansion in powers of the eccentricity was obtained recently [2]. In this section, we find the distribution for this case in terms of elementary functions.

When $\mu = \lambda$, expression (8) becomes

$$\rho_{\lambda,\lambda}(\ell) = \int_0^{\frac{\pi}{2}} d\theta \sin\theta \frac{1}{\sqrt{\lambda^2\sin^2\theta + \cos^2\theta}}\rho\left(\frac{\ell}{\sqrt{\lambda^2\sin^2\theta + \cos^2\theta}}\right) = \int_0^1 dx \frac{1}{\sqrt{\lambda^2(1-x^2)+x^2}}\rho\left(\frac{\ell}{\sqrt{\lambda^2(1-x^2)+x^2}}\right). \tag{9}$$

To do the integration we redefine ρ as

$$\rho(x) = \frac{3}{16}(x-2)^2x^2(x+4) \text{ for } \textit{all} \text{ x}. \tag{10}$$

Then:

$$\rho_{\lambda,\lambda}(\ell) = \begin{cases} \int_0^1 dx \frac{1}{\sqrt{\lambda^2+(1-\lambda^2)x^2}}\rho\left(\frac{\ell}{\sqrt{\lambda^2+(1-\lambda^2)x^2}}\right) & \text{when } 0 < \ell < 2 \\ \int_0^{\frac{1}{2}\sqrt{\frac{4\lambda^2-\ell^2}{\lambda^2-1}}} dx \frac{1}{\sqrt{\lambda^2+(1-\lambda^2)x^2}}\rho\left(\frac{\ell}{\sqrt{\lambda^2+(1-\lambda^2)x^2}}\right) & \text{when } 2 < \ell < 2\lambda \end{cases} \tag{11}$$

so that the argument of ρ is always smaller than 2. Both integrals can be expressed in terms of elementary functions:

The first integral (valid when $\ell < 2$) is

$$12\left(\frac{\ell}{2\lambda}\right)^2 - 9\left(\lambda + \frac{\ln(\lambda+\sqrt{\lambda^2-1})}{\sqrt{\lambda^2-1}}\right)\left(\frac{\ell}{2\lambda}\right)^3 + \frac{3}{4}\left(\lambda(3+2\lambda^2) + 3\frac{\ln(\lambda+\sqrt{\lambda^2-1})}{\sqrt{\lambda^2-1}}\right)\left(\frac{\ell}{2\lambda}\right)^5 \tag{12}$$

and the second one (valid when $2 < \ell < 2\lambda$) is

$$\frac{9\ell}{128\lambda^5\sqrt{\lambda^2-1}}\left[2\lambda\sqrt{4\lambda^2-d^2}(8\lambda^2+\ell^2) + \ell^2(\ell^2-16\lambda^2)\ln\left(\frac{2\lambda+\sqrt{4\lambda^2-\ell^2}}{d}\right)\right]. \tag{13}$$

It is obvious that the first integral can be simplified if written in terms of the variable $a \equiv \ell/2\lambda$. The second integral can be simplified if written in terms of the variable $b \equiv (\ell/2\lambda)^2$:

$$\frac{9\sqrt{b}}{4\sqrt{\lambda^2-1}}\left[\sqrt{1-b}(2+b) + b(b-4)\ln\left(\frac{1+\sqrt{1-b}}{\sqrt{b}}\right)\right]. \tag{14}$$

5. A spheroidal approximation to the shape of the earth

A non-rotating mass adopts a spherical shape due to gravitation. When the mass rotates, the centrifugal force pushes the mass away form the axis of rotation. In equilibrium these two forces balance. In terms of a potential, the configuration of equilibrium will minimize the potential. In this section, we are going to find this configuration of equilibrium for a self-gravitating fluid of constant density which rotates without differential rotation and adopts an ellipsoidal shape.

The expected value of $1/\ell$ of a distribution of mass or charge is, up to a constant, its gravitational or electrostatic potential energy. For the spheroid of homogeneous density we can compute it from formula (11) or, better yet, from the expressions (12) and (14). These two contributions are, respectively:

$$\int_0^2 \mathrm{d}\ell \frac{\rho(\ell)}{\ell} = \frac{3\lambda\sqrt{\lambda^2-1}(3+22\lambda^2) + (9-60\lambda^2)\operatorname{arg\,tanh}\frac{\sqrt{\lambda^2-1}}{\lambda}}{20\lambda^5\sqrt{\lambda^2-1}}, \tag{15}$$

and

$$\int_2^{2\lambda} \mathrm{d}\ell \frac{\rho(\ell)}{\ell} = \frac{-3\lambda\sqrt{\lambda^2-1}(3+22\lambda^2) - (9-60\lambda^2)\operatorname{arg\,tanh}\frac{\sqrt{\lambda^2-1}}{\lambda}}{20\lambda^5\sqrt{\lambda^2-1}} + \frac{3\pi - 6\arcsin\frac{1}{\lambda}}{5\sqrt{\lambda^2-1}}, \tag{16}$$

which add up to

$$\frac{3\pi - 6\arcsin\frac{1}{\lambda}}{5\sqrt{\lambda^2-1}}. \tag{17}$$

To obtain the gravitational potential energy of a spheroid whose short semiaxis is R and whose long semiaxes are λR, we multiply the above expression by $\frac{1}{2}M^2\frac{G}{R}$. The factor $\frac{1}{2}$ prevents overcounting, the mass squared M^2 takes care of the normalization, G is the gravitational constant and $\frac{1}{R}$ measures the distance in the appropriate units. The result is

$$V_{\mathrm{gr}}(\lambda) = M^2\frac{G}{R}\frac{\left(3\pi - 6\arcsin\frac{1}{\lambda}\right)}{10\sqrt{\lambda^2-1}}. \tag{18}$$

One can check, using l'Hôpital's rule, that the above expression yields the correct result, $\frac{3}{5}M^2\frac{G}{R}$, for the gravitational self-energy of a sphere.

Suppose that a self-gravitating fluid of constant density ρ rotates without differential rotation and adopts an ellipsoidal shape. Which one would it be? Due to the symmetry of the problem we assume that the searched ellipsoid is a spheroid whose short ('vertical') semiaxis measures R and whose two other semiaxes measure λR. The constraint

$$M = \rho\tfrac{4}{3}\pi R^3\lambda^2 \tag{19}$$

implies

$$R = \sqrt[3]{\frac{3M}{4\pi\rho\lambda^2}}. \tag{20}$$

Substitution of this constraint into formula (18) yields

$$V_{\text{gr}}(\lambda) = GM^{5/3}\frac{\left(3\pi - 6\arcsin\frac{1}{\lambda}\right)((4/3)\pi\rho)^{1/3}\lambda^{2/3}}{10\sqrt{\lambda^2-1}}. \tag{21}$$

We also have to take the centrifugal acceleration, $\omega^2 r$ into account. It is convenient to pretend that the centrifugal force has its origin in a potential equal to the mass times $\omega^2 r^2/2$. Then, the centrifugal potential energy of a disc of radius r is

$$\int_0^r \mathrm{d}r'\, 2\pi r'\frac{\rho}{2}\omega^2 r'^2 = \frac{\pi\rho\omega^2}{4}r^4. \tag{22}$$

For a spheroid whose equation in cylindrical coordinates is $(z/R)^2 + (r/\lambda R)^2 = 1$, its centrifugal potential energy is

$$\frac{\pi\rho\omega^2}{4}\lambda^4\int_{-R}^{R}\mathrm{d}z(R^2-z^2)^2 = \frac{\pi\rho\omega^2}{4}\lambda^4\frac{16}{15}R^5 = \frac{M\omega^2\lambda^2R^2}{5}. \tag{23}$$

Substitution of the constraint (20) into the above formula yields

$$V_{\text{centr}}(\lambda,\omega) = \frac{M^{5/3}\omega^2\lambda^{2/3}}{5\left((4/3)\pi\rho\right)^{2/3}}. \tag{24}$$

For a given angular velocity the gravitational potential energy increases as the fluid becomes more oblate, while the centrifugal potential energy diminishes. There must be then an equilibrium oblateness. To find it we set the derivative with respect to λ of the potential

$$V(\lambda,\omega) \equiv V_{\text{gr}}(\lambda) + V_{\text{centr}}(\lambda,\omega) \tag{25}$$

equal to zero. One can solve analytically for ω:

$$\frac{\omega^2}{G\pi\rho} = \frac{(2+\lambda^2)\left(\pi - 2\arcsin\frac{1}{\lambda}\right)}{(\lambda^2-1)^{3/2}} - \frac{6}{\lambda^2-1}. \tag{26}$$

This formula was found by Maclaurin in 1742, who obtained it from the condition of hydrostatic equilibrium of two hypothetical columns of fluid, one polar and the other equatorial, who join at the centre of the Earth [24], pages 3 and 78.

Acknowledgments

Had the architect Michael L Benedikt not suggested that I study distributions of distance, this paper would not have been written.

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